Dynamic Stability of a Flexible Missile under Constant and Pulsating Thrusts

T. R. BEAL*
Lockheed Missiles and Space Company, Huntsville, Ala.

The stability of a flexible missile, idealized as a uniform free beam under an end thrust, is investigated. A simplified control system is incorporated to obtain directional stability. It is shown that, in the absence of the control system, the critical thrust magnitude is associated with coalescence of the two lowest bending frequencies. With the control system included, it is found that the critical thrust magnitude corresponds to a reduction of the lowest frequency to zero. The results are related to typical thrust levels found in large, present-day missiles. Also considered is the effect of periodically varying thrust perturbations on the stability of the vehicle. In this case, parametric instabilities are found to exist. It is found that, when the beam is very stiff longitudinally, instabilities are most likely to occur when the frequency of the thrust variation is in the vicinity of 1) twice any of the bending frequencies, 2) the sum of any two of the bending frequencies, or 3) the difference of any two of the bending frequencies. With low longitudinal compliance, significant instabilities are also found to occur when the frequency of the thrust variation is in the vicinity of one of the longitudinal natural frequencies.

Nomenclature

A = beam cross-sectional area
$c_k^{(m)}$ = the kth element in the matrix $[c_k]^{(m)}$
E = elastic modulus
EI = bending stiffness of uniform beam
F_{jk} = coefficient in the jth row and kth column of matrix $[F_{jk}]$
G_{jk} = coefficient in the jth row and kth column of matrix $[G_{jk}]$
K_θ = directional control factor determining thrust vector gimbal angle
l = length of uniform beam
m = mass per unit length of beam
N = number of bending degrees of freedom assumed in numerical analysis
p = lateral force on beam arising from component of thrust due to gimbaling
P = axial force distribution in beam
q_A, q_B = rigid-body generalized coordinates
q_n = nth bending generalized coordinate
R_1 = modulus of $Z + (Z^2 - 1)^{1/2}$
R_2 = modulus of $Z - (Z^2 - 1)^{1/2}$
t = time
T_0 = amplitude of constant thrust
T_1 = amplitude of sinusoidally varying thrust
$\bar{T}_0$ = nondimensional thrust parameter $= T_0 l^2/EI$
$u(x, t)$ = longitudinal displacement of particles of beam measured in Lagrangian coordinate system
x = Lagrangian coordinate defining position of particles in unstrained beam relative to one end of the beam
x_G = x coordinate corresponding to the location of direction-sensing element in the beam
$y(x, t)$ = lateral displacement of axis of beam from fixed reference line
Z = $\cos 2\pi\alpha$
α = characteristic exponent whose value indicates the stability of a system whose motion is represented by differential equations with periodic coefficients
β_1 = argument of $Z + (Z^2 - 1)^{1/2}$
β_2 = argument of $Z - (Z^2 - 1)^{1/2}$
γ = T_1/T_0
$\delta(\xi)$ = Dirac delta function
θ = gimbal angle, equal to rotation of thrust vector from a tangent to the beam-deflection curve
λ_n^4 = uniform beam frequency parameter $= \omega_n^2(ml^4/EI)$
ξ = nondimensional coordinate $= x/l$
τ = nondimensional time variable $= \omega_1 t$
$\bar{\sigma}$ = Ω/ω_L
ϕ_n = mode shape of nth vibration mode of uniform free-free beam
$\Phi(\xi)$ = function defining the longitudinal force distribution in a uniform beam arising from the varying thrust component
Ψ_G = rotation of the beam element located at x_G
$[\Psi_k(\tau)]$ = matrix whose elements have a periodic variation of 2π in τ
ω_L = fundamental longitudinal frequency of free-free beam
ω_n = lateral bending frequency of nth mode of free-free beam
$\omega_{(n)}$ = lateral bending frequency of nth mode of free-free beam with end thrust
$\bar{\omega}_L$ = nondimensional longitudinal frequency $= \omega_L/\omega_1$
$\bar{\omega}_n$ = nondimensional bending frequency $= \omega_n/\omega_1$
$\bar{\omega}_{(n)}$ = nondimensional bending frequency $= \omega_{(n)}/\omega_1$
Ω = frequency of thrust variation
$\bar{\Omega}$ = nondimensional frequency of thrust variation $= \Omega/\omega_1$
$(\dot{\ })$ = $d/d\tau$ or $\partial/\partial\tau$
$(\ )'$ = $d/d\xi$ or $\partial/\partial\xi$

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* Research Specialist, Huntsville Research and Engineering Center. Member AIAA.

Introduction

A CURRENT trend in the development of missiles is in the direction of more flexible vehicles, the degree of flexibility being defined qualitatively (and somewhat nebulously) as the extent to which bending characteristics are significant in the dynamic analysis of the vehicle. Corresponding to this trend is an increase in the significance of the longitudinal inertia forces (due to the vehicle's thrust) on the bending vibrations of the missile. When one becomes concerned about such an effect, one must also be concerned about the stability of the vehicle. The question may be asked: will the missile buckle or perhaps be afflicted with an oscillatory instability? Even if the answer to such questions is negative, one would still be interested in the effects of longitudinal acceleration if such effects appreciably alter the frequencies of bending vibrations.

The problem being considered in this paper applies most generally to the dynamic stability of a flexible missile under an end thrust of magnitude $T_0 + T_1 \cos\Omega t$, where T_0, T_1, and Ω are constants and t is time. The special case of stability under constant thrust is obtained by setting $T_1 = 0$. Simple beam theory is used in describing the bending characteristics of the vehicle, and, for simplicity, uniform mass and stiffness distributions are assumed. The direction of the thrust is assumed to be controlled by means of a feedback system that produces a linear relationship between the gimbal angle θ (see Fig. 1) and the rotation ψ_G of the element at some predetermined location on the vehicle.

Such a model represents an idealization of a slender, flexible rocket vehicle with directional control, having its engine thrust subjected to periodic variations in its magnitude. These fluctuations may result from variations in the rate at which liquid fuel is fed into the combustion chamber, or possibly from periodic variations of the flow pattern of the gases passing through the exhaust nozzle. To an even larger degree, such periodic fluctuations are inherent in the operation of pulse-jet engines.

An actual missile has, in general, highly nonuniform mass and stiffness distribution, structural damping, and mass transfer characteristics as well as servo lag effects, engine inertia, and rate feedback control. All of these effects are neglected in this analysis, as are aerodynamic forces and the effects of shear and rotary inertia. Because of this high degree of simplification, it is emphasized that the results obtained represent conditions that may be typical of similar conditions existing for an actual missile, but, except in a very general sense, are not quantitatively applicable to a specific missile. For this reason an exhaustive study showing the effects of all the pertinent parameters is not attempted, but rather certain typical cases that would seem to be the most instructive are investigated.

Analytical Development

The beam whose stability is to be investigated is shown in Fig. 1. Displacements of the particles of the beam are defined relative to a Lagrangian coordinate system in which x defines a location on the beam in some initial unstrained state. It is assumed that in this reference state the beam is fixed relative to a Newtonian frame of reference and that its axis coincides with a fixed reference line from which lateral motion will be measured during the actual motion of the beam. Two-dimensional motion is assumed. Gravitational forces are neglected for convenience, although for purposes of predicting stability, the results of this investigation may be considered applicable to cases where such forces actually exist.

Displacements of points lying on the middle surface of the beam in a direction parallel to the reference line are denoted by $u(x, t)$, the positive direction being chosen to coincide with the positive x direction. Similarly, the displacements perpendicular to the reference line are denoted by $y(x, t)$, the positive direction being as indicated in the figure.

Assuming simple beam theory, one may readily derive the following well-known equations:

$$\partial^2 u/\partial x^2 = (m/AE)(\partial^2 u/\partial t^2) \tag{1}$$

$$EI\,\frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x}\left(P\,\frac{\partial y}{\partial x}\right) + m\,\frac{\partial^2 y}{\partial t^2} - p = 0 \tag{2}$$

where m is mass per unit length, A the cross-sectional area, EI the bending stiffness, $P(x, t)$ the longitudinal force, and $p(x, t)$ represents the lateral load (per unit length) other than the inertia loading. The validity of Eqs. (1) and (2) is based on the assumption that $\partial y/\partial x$ and $\partial u/\partial x$ are small in comparison to unity. Consistent with this assumption, and the additional assumption that the gimbal angle θ is small, the

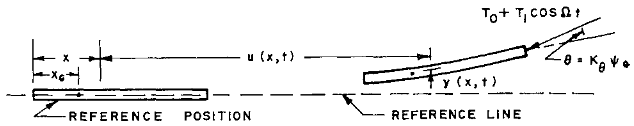

Fig. 1 Displacements in Lagrangian coordinate system.

following boundary conditions on the longitudinal displacement u must be satisfied:

$$\begin{aligned} \partial u/\partial x &= 0 \text{ at } x = 0 \\ \partial u/\partial x &= (T_0/AE)\,(1 + \gamma \cos\Omega t) \text{ at } x = l \end{aligned} \tag{3}$$

where $\gamma = T_1/T_0$.

It may be verified by direct substitution that the following particular solution satisfies the differential equation (1) and the boundary conditions of Eq. (3):

$$u = -\frac{T_0 x^2}{2AEl} - \frac{T_0 t^2}{2ml} + \frac{T_1}{\Omega}\,\frac{1}{(mAE)^{1/2}}\,\frac{\cos[(m/AE)^{1/2}\Omega x]}{\sin[(m/AE)^{1/2}\Omega l]}\cos\Omega t \tag{4}$$

This solution represents the steady-state response to the applied end thrust $T_0 + T_1 \cos\Omega t$ and neglects the transient response of the longitudinal natural modes.

From the relation $P = -AE(\partial u/\partial x)$ it follows that the longitudinal force distribution in the beam is expressed by

$$P(\xi,t) = T_0[\xi + \gamma\Phi(\xi)\cos\Omega t] \tag{5}$$

where ξ is the nondimensional variable $\xi = x/l$ and

$$\Phi(\xi) = \frac{\sin[(m/AE)^{1/2} l\Omega\xi]}{\sin[(m/AE)^{1/2} l\Omega]} \tag{6}$$

The formula for the fundamental frequency of longitudinal vibrations of a uniform free-free beam is[1]

$$\omega_L = (\pi/l)(AE/m)^{1/2} \tag{7}$$

Thus, Eq. (6) may be written in the form

$$\Phi(\xi) = \frac{\sin(\pi\xi\Omega/\omega_L)}{\sin(\pi\Omega/\omega_L)} \tag{8}$$

Note that if the ratio of the forcing frequency Ω to the fundamental longitudinal frequency ω_L is small, then $\Phi(\xi) \simeq \xi$. In such a case, the axial force distribution is linear with ξ, indicating a rigid-body longitudinal response. On the other hand, if the forcing frequency Ω is in the neighborhood of ω_L (or an integral multiple of ω_L), longitudinal resonance occurs and P assumes very large values for intermediate values of ξ.

It is convenient to consider the thrust force in terms of two components, one in the direction of the tangent to the deflection curve with the approximate magnitude $T_0 + T_1 \cos\Omega t$ and the other perpendicular to this tangent with the approximate magnitude $(T_0 + T_1 \cos\Omega t)\theta$. On the basis of the assumption that $\partial y/\partial x$ and $\partial u/\partial x$ are both small, θ may be replaced by $\theta = K_\theta[\partial y/\partial x(x_G,t)]$. The component $(T_0 + T_1 \cos\Omega t)\theta$ is considered as an external force applied to the beam to be included in Eq. (2) as p.

If we substitute Eq. (5) into Eq. (2), express p in terms of the forementioned component of the thrust force, and introduce nondimensional variables $\xi = x/l$ and $\tau = \omega_1 t$ (where ω_1 is the fundamental bending frequency of the uniform free-free beam), Eq. (2) becomes

$$\frac{\partial^4 y}{\partial \xi^4} + \bar{T}_0\,\frac{\partial}{\partial \xi}\left[\frac{\partial y}{\partial \xi}\,(\xi + \gamma\Phi\cos\bar{\Omega}\tau)\right] + \lambda_1^4\,\frac{\partial^2 y}{\partial \tau^2} + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)\,K_\theta\,\frac{\partial y}{\partial \xi}\,(\xi_G,\tau)\,\delta(\xi - 1) = 0 \tag{9}$$

where $\bar{T}_0 = T_0 l^2/EI$, $\bar{\Omega} = \Omega/\omega_1$, $\lambda_1^4 = \omega_1^2(ml^4/EI)$, and $\delta(\xi)$ is the Dirac delta function.

Having considered the component $(T_0 + T_1 \cos\Omega t)\theta$ of the thrust force in the differential equation, we need consider only the tangential component of the thrust force in establishing the boundary conditions. From consideration of the equilibrium of the basic beam element and the kinematics of the stress-strain relationship, one may deduce the following boundary conditions:

$$\begin{aligned} \xi &= 0 \qquad \frac{\partial^2 y}{\partial \xi^2} = 0 \qquad \frac{\partial^3 y}{\partial \xi^2} = 0 \\ \xi &= 1 \qquad \frac{\partial^2 y}{\partial \xi^2} = 0 \qquad \frac{\partial^3 y}{\partial \xi^3} = 0 \end{aligned} \tag{10}$$

Note that the only restriction placed on the displacements is that $\partial y/\partial x$ and $\partial u/\partial x$ should both remain small in comparison to unity. Obviously, the displacements $u(x, t)$ themselves become large. There is no reason why $y(x, t)$ should not be allowed to become large also, provided the slope $[\partial y/\partial x(x, t)]$ remains small. In fact, we note that there is actually no method of controlling the lateral displacement and that a constant lateral velocity, as well as a constant lateral displacement, is possible. This permits arbitrarily large values of $y(x, t)$ to result.

Equation (9) is a linear partial differential equation in the dependent variable y and the independent variables ξ and τ. A series solution may be obtained by expressing the deflection (assuming small slopes) as

$$y(\xi, \tau) = q_A(\tau) + q_B(\tau)\xi + \sum_{n=1}^{\infty} q_n(\tau)\phi_n(\xi) \tag{11}$$

where $q_A(\tau)$ and $q_B(\tau)$ are rigid-body coordinates and $q_n(\tau)$ the coordinate associated with the function $\phi_n(\xi)$. This function is assumed to be the nth vibration mode shape of the free-free beam with no thrust. Thus, $\phi_n(\xi)$ satisfies the differential equation

$$d^4\phi_n/d\xi^4 = \lambda_n{}^4\phi_n \qquad \lambda_n{}^4 = \omega_n{}^2(ml^4/EI) \tag{12}$$

and the boundary conditions

$$\begin{aligned} \xi &= 0 \qquad \frac{d^2\phi_n}{d\xi^2} = 0 \qquad \frac{d^3\phi_n}{d\xi^3} = 0 \\ \xi &= 1 \qquad \frac{d^2\phi_n}{d\xi^2} = 0 \qquad \frac{d^3\phi_n}{d\xi^3} = 0 \end{aligned} \tag{13}$$

ω_n being the natural frequency of the nth mode of vibration of the free-free beam.

Observe that all the boundary conditions of Eq. (10) are satisfied by each of the functions in Eq. (11). With this condition satisfied, the Galerkin method offers a useful means of obtaining an approximation to the solution when a finite number of terms in the series is assumed.[2-4] This method converts Eq. (9) into a set of ordinary differential equations.

The expression

$$y_N(\xi,\tau) = q_A(\tau) + q_B(\tau)\xi + \sum_{n=1}^{N} q_n(\tau)\phi_n(\xi) \tag{14}$$

when substituted into Eq. (9) leads inevitably to a certain error. In the Galerkin procedure this error is weighted by each of the approximating functions, and its integral over the length of the beam is set equal to zero, thereby leading to the following equations:

$$\int_0^1 \left\{ \frac{\partial^4 y_N}{\partial \xi^4} + \bar{T}_0 \frac{\partial}{\partial \xi}\left[\frac{\partial y_N}{\partial \xi}(\xi + \gamma\Phi\cos\bar{\Omega}\tau)\right] + \lambda_1{}^4 \frac{\partial^2 y_N}{\partial \tau^2} + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \times \frac{\partial y_N}{\partial \xi}(\xi_G)\delta(\xi - 1)\right\} d\xi = 0$$

Continued in next column

$$\int_0^1 \left\{ \frac{\partial^4 y_N}{\partial \xi^4} + \bar{T}_0 \frac{\partial}{\partial \xi}\left[\frac{\partial y_N}{\partial \xi}(\xi + \gamma\Phi\cos\bar{\Omega}\tau)\right] + \lambda_1{}^4 \frac{\partial^2 y_N}{\partial \tau^2} + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \times \frac{\partial y_N}{\partial \xi}(\xi_G)\delta(\xi - 1)\right\} \xi d\xi = 0$$

$$\int_0^1 \left\{ \frac{\partial^4 y_N}{\partial \xi^4} + \bar{T}_0 \frac{\partial}{\partial \xi}\left[\frac{\partial y_N}{\partial \xi}(\xi + \gamma\Phi\cos\bar{\Omega}\tau)\right] + \lambda_1{}^4 \frac{\partial^2 y_N}{\partial \tau^2} + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \times \frac{\partial y_N}{\partial \xi}(\xi_G)\delta(\xi - 1)\right\} \phi_k(\xi) d\xi = 0$$

$$k = 1, 2, 3, \ldots, N \tag{15}$$

As a result of the equilibrium conditions and the orthogonality properties of the modes of vibration of a uniform free-free beam, we have

$$\begin{aligned} &\int_0^1 \phi_n(\xi)\, d\xi = 0 \qquad \int_0^1 \xi\phi_n(\xi) d\xi = 0 \\ &\int_0^1 \phi_m(\xi)\phi_n(\xi) d\xi = 0 \text{ if } m \neq n \end{aligned} \tag{16}$$

The actual magnitude of $\phi_n(\xi)$ is arbitrary to a constant factor. We assume that this factor is such that

$$\int_0^1 \phi_n{}^2 d\xi = 1 \tag{17}$$

Substituting Eq. (14) into Eq. (15) and utilizing Eqs. (12, 13, 16, and 17), we obtain, after performing the necessary integrations, the following system of ordinary differential equations:

$$\left.\begin{aligned}
&\lambda_1{}^4\ddot{q}_A + \frac{1}{2}\lambda_1{}^4\ddot{q}_B + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau) \times \left[q_B + \sum_{n=1}^{N} q_n\phi_n{}'(1)\right] + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \times \left[q_B + \sum_{n=1}^{N} q_n\phi_n{}'(\xi_G)\right] = 0 \\
&\frac{\lambda_1{}^4}{2}\ddot{q}_A + \frac{\lambda_1{}^4}{3}\ddot{q}_B + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau) \times \left[q_B + \sum_{n=1}^{N} q_n\phi_n{}'(1)\right] - \frac{\bar{T}_0}{2} q_B - \bar{T}_0\gamma\cos\bar{\Omega}\tau \times \left(\frac{1 - \cos\pi\bar{\sigma}}{\pi\bar{\sigma}\ \sin\pi\bar{\sigma}}\right) q_B - \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau) \times \\
&\quad \sum_{n=1}^{N} q_n\phi_n(1) + \bar{T}_0\gamma\cos\bar{\Omega}\tau \sum_{n=1}^{N} q_n \int_0^1 \phi_n\,\Phi' d\xi + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta\left[q_B + \sum_{n=1}^{N} q_n\phi_n(\xi_G)\right] = 0 \\
&\lambda_1{}^4\ddot{q}_k + \lambda_k{}^4 q_k + \bar{T}_0(1 + \gamma\cos\Omega\tau) \times \left[q_B + \sum_{n=1}^{N} q_n\phi_n{}'(1)\right]\phi_k(1) - \bar{T}_0\phi_k(1)q_B - \\
&\quad \bar{T}_0\gamma\cos\bar{\Omega}\tau\left[\phi_k(1) - \int_0^1 \phi_k\,\Phi' d\xi\right] q_B - \bar{T}_0 \sum_{n=1}^{N} q_n \int_0^1 \xi\phi_n{}'\phi_k{}' d\xi - \\
&\quad \bar{T}_0\gamma\cos\bar{\Omega}\tau \sum_{n=1}^{N} q_n \int_0^1 \Phi\phi_n{}'\phi_k{}' d\xi + \bar{T}_0(1 + \gamma\cos\bar{\Omega}\tau)K_\theta\left[q_B + \sum_{n=1}^{N} q_n\phi_n{}'(\xi_G)\right]\phi_k(1) = 0 \\
&\qquad k = 1, 2, \ldots N
\end{aligned}\right\} \tag{18}$$

In the foregoing equations, $\alpha = \Omega/\omega_L$, a prime indicates $d/d\xi$, and a dot indicates $d/d\tau$.

The first two equations may be combined in such a way as to eliminate q_A. Doing this, and expressing the resulting equations in matrix form, we obtain

$$[\ddot{q}_k] + [F_{jk}][q_k] + \gamma \cos\bar{\Omega}\tau[G_{jk}][q_k] = 0 \qquad (19)$$

where

$$[q_k] = \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_N \\ q_B \end{bmatrix}$$

and $[F_{jk}]$ and $[G_{jk}]$ are square matrices of order $N + 1$. The elements of $[F_{jk}]$ and $[G_{jk}]$ are defined as follows:

$$\left.\begin{aligned}
F_{jk} &= \bar{\omega}_j{}^2\delta_{jk} + \frac{\bar{T}_0}{\lambda_1{}^4} \times \\
&\left[\phi_j(1)\phi_k'(1) - \int_0^1 \xi\phi_j'\phi_k'd\xi + K_\theta\phi_j(1)\phi_k'(\xi_G)\right] \\
&\qquad j = 1, 2, \ldots N \qquad k = 1, 2, \ldots N \\
F_{j,N+1} &= \bar{T}_0 K_\theta \phi_j(1)/\lambda_1{}^4 \qquad j = 1, 2, \ldots N \\
F_{N+1,k} &= (12\,\bar{T}_0/\lambda_1{}^4)\,[\tfrac{1}{2}\phi_k'(1) - \phi_k(1) + \tfrac{1}{2}K_\theta\phi_k'(\xi_G)] \\
&\qquad k = 1, 2, \ldots N \\
F_{N+1,N+1} &= 6\bar{T}_0 K_\theta/\lambda_1{}^4 \\
G_{jk} &= \frac{\bar{T}_0}{\lambda_1{}^4}\left[\phi_j(1)\phi_k'(1) - \int_0^1 \Phi\phi_j'\phi_k'd\xi + K_\theta\phi_j(1)\phi_k'(\xi_G)\right] \\
&\qquad j = 1, 2, \ldots N \qquad k = 1, 2, \ldots N \\
G_{j,N+1} &= \frac{\bar{T}_0}{\lambda_1{}^4}\left[\int_0^1 \phi_j'\Phi'd\xi + K_\theta\phi_j(1)\right] \\
&\qquad j = 1, 2, \ldots N \\
G_{N+1,k} &= \frac{12\bar{T}_0}{\lambda_1{}^4}\left[\tfrac{1}{2}\phi_k'(1) - \phi_k(1) + \int_0^1 \phi_k\Phi'd\xi + \tfrac{1}{2}K_\theta\phi_k'(\xi_G)\right] \qquad k = 1, 2, \ldots N \\
G_{N+1,N+1} &= \frac{12\bar{T}_0}{\lambda_1{}^4}\left[\frac{1}{2} - \frac{1 - \cos\pi\bar{\sigma}}{\pi\bar{\sigma}\sin\pi\bar{\sigma}} + \frac{1}{2}K_\theta\right]
\end{aligned}\right\} \qquad (20)$$

where $\bar{\omega}_j = \omega_j/\omega_1$ and δ_{jk} is the Kronecker delta ($\delta_{jk} = 1$ if $j = k$, $\delta_{jk} = 0$ if $j \neq k$).

Equation (19) constitutes a set of ordinary, linear differential equations having, in general, periodically varying coefficients. The stability of the vibratory motion of the beam is determined by considering the nature of the solutions of this set of equations. Special cases are of interest and may be investigated by prescribing particular values for the various parameters involved.

Part I: Stability under Constant Thrust

The problem of the stability of the beam under a thrust of constant magnitude is of interest in itself. The equations defining the motion of such a system are obtained by setting $\gamma = 0$ in Eq. (19). In this case the periodicity of the coefficients vanishes. The characteristic modes of motion of the system so defined may be determined by representing q_k in the form

$$[q_k] = [\bar{q}_k]e^{i\bar{\omega}\tau} \qquad (21)$$

in which $\bar{\omega} = \omega/\omega_1$ (ω being the frequency with respect to the real-time variable t), and $[\bar{q}_k]$ is a column matrix of constants. Substituting Eq. (21) into (19) (with $\gamma = 0$), we obtain a set of linear, homogeneous, algebraic equations. To obtain nontrivial solutions for these equations, we must set the determinant of coefficients equal to zero; that is,

$$\det\{[F_{jk}] - \bar{\omega}^2\,[I]\} = 0 \qquad (22)$$

where I is the identity matrix.

The characteristic equation, obtained by expanding the determinant represented in Eq. (22) and setting it equal to zero, is a polynomial in $\bar{\omega}^2$. For the beam to be stable in every mode, it is necessary that the values of $\bar{\omega}^2$, obtained as roots of the characteristic equation, be real and positive. Negative or complex values for $\bar{\omega}^2$ indicate the existence of at least one unstable mode.

Stability without Control System ($K_\theta = 0$)

If $K_\theta = 0$, we have the case of a beam under tangential end thrust. It is observed by a detailed examination of Eq. (18) (it is also apparent from physical considerations) that, if K_θ is zero, two zero-frequency modes exist for all values of $\bar{T}_0$. The characteristic motion associated with one of these zero-frequency modes is lateral translation of the beam with no accompanying rotation or bending. The other mode involves rotation of the beam at a constant angular velocity (or merely a constant angular displacement) accompanied by translation; but again no bending is involved. These modes are, in the strictest sense, unstable, so that the system is unstable a priori, no matter whether the vibratory modes are stable or not. However, since no bending of the beam is involved, structural failure does not occur as a result of motion in these modes. Thus, defining a critical thrust magnitude as the magnitude at which one of the vibratory modes has impending instability is justified.

There exists, however, one additional problem. If there is motion in the zero-frequency rotational mode, large slopes $\partial y/\partial x$ will occur. The appearance of these large slopes violates one of the assumptions on which the derivation of the equations of motion was based. Thus, to get meaningful solutions for this case, we may impose the restriction that there be no motion in the zero-frequency rotational mode. An alternative point of view (and a more physically satisfying one) is to assume that, if motion does exist in this mode, the reference line from which the displacements $y(x, t)$ are measured rotates with the same angular velocity as that of the rigid-body rotation in the zero-frequency rotational mode. Thus, the original assumption of small $\partial y/\partial x$ is valid, and the derivation of the equations of motion remains unchanged if the angular velocity of rotation is small enough that the centrifugal and Coriolis forces associated with the rotating coordinate system may be neglected.

The equations for the case $K_\theta = 0$ were programed for solution on the IBM 7090 computer. Solutions for the characteristic frequencies were obtained for progressively larger values of $\bar{T}_0$ until an unstable root was obtained. The analysis was repeated for several different numbers of bending degrees of freedom, (i.e., several different values of N) to determine how many modes are necessary to represent adequately the frequency trend. If we use a value of $N = 1$, the fundamental (and only) frequency reduces to zero at a value of $\bar{T}_0 = 81.4$. For $N = 2$ and higher, the first unstable root occurs as a result of the coalescence of the first and second characteristic frequencies. The variation of these frequencies as $\bar{T}_0$ is increased is shown in Fig. 2 for values of N equal to 1, 2, 3, 4, and 5. The difference between the curves for $N > 2$ cannot be detected when the curves are plotted to the scale shown in the figure.

The critical value of $\bar{T}_0$ for $N = 5$ was found to be $\bar{T}_{cr} = 109.9$, which differs from the value found for $N = 4$ by less than 0.1%. Thus, within 0.1% accuracy, we may assume the critical load to be equal to 109.9. This value is compared in

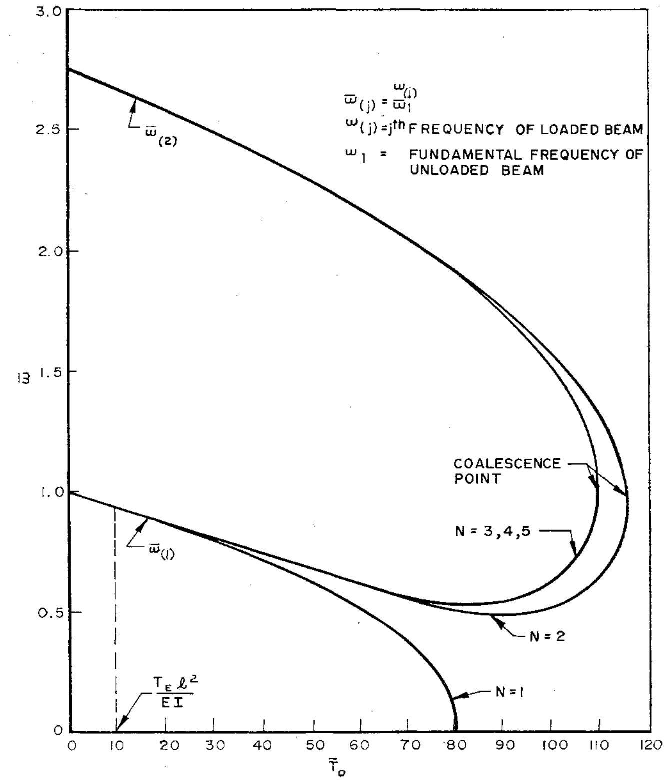

Fig. 2 Variation of frequency with thrust for several values of N ($K_\theta = 0$).

Fig. 2 with the Euler buckling load $T_E = \pi^2EI/l^2$. It is seen that the critical load for the problem investigated here is approximately eleven times the Euler buckling load.

Higher mode stability was investigated for $\bar{T}_0 > \bar{T}_{cr}$ by solving for the characteristic frequencies and noting their trend. Eight bending modes were used in the analysis. Figure 3 shows that coalescence of the frequencies corresponding to the third and fourth bending modes occurs for $\bar{T}_0 =$ 405, whereas the frequencies corresponding to the next two higher modes coalesce for $\bar{T}_0 = 870$. Based on this trend, it seems reasonable to expect all of the higher modes of instability to be created by pairwise coalescence of the characteristic frequencies of the system.

Stability with Control System ($K_\theta > 0$)

Solutions for the characteristic frequencies were obtained on the 7090 computer for several values of K_θ and ξ_G. Recall that, when $K_\theta = 0$, two zero-frequency modes existed for all values of $\bar{T}_0$. In the case $K_\theta > 0$, we anticipate that in general there exists only one zero-frequency solution, namely, one that corresponds to the lateral translation of the beam. Rigid-body rotation is no longer expected to be a zero-frequency solution because of the characteristics of the control system. We may conclude, as in the case without a control system, that, because of the existence of the zero-frequency translational mode, the beam is unstable a priori. However, since no bending of the beam is involved in this mode, we are justified in considering, as before, the critical thrust magnitude to be that at which one of the vibratory modes has impending instability.

The frequencies were computed for a range of values of K_θ and ξ_G using two bending degrees of freedom in addition to the two rigid-body degrees of freedom. In Fig. 4 curves of frequency vs thrust are shown. One frequency curve, not evident in the figure, lies along the $\bar{\omega} = 0$ axis and corresponds to the rigid-body translation mode. In addition to the frequencies shown in the figure, a higher frequency existed in each case, arising from the second bending degree of freedom. The use of a larger number of bending degrees of freedom in the analysis would introduce even more frequency curves as well as alter slightly the lower frequency curves shown in Fig. 4. Qualitatively, however, these lower frequency curves would be expected to have the same general behavior as shown in Fig. 4. It is not anticipated, therefore, that the inclusion of a larger number of bending degrees of freedom would effectively alter the stability of the system.

Note that in curves a and b of Fig. 4 (corresponding to values of $\xi_G = 0$ and $\xi_G = 0.2$) an unstable region occurs for $K_\theta = 1$ as a result of frequency coalescence. For larger values of ξ_G ($\xi_G = 0.5$ and $\xi_G = 0.8$, for example) this region of instability does not occur, at least for the range of values of K_θ considered. Furthermore, from the trend (with K_θ) of the upper sets of frequency curves in c and d, it does not appear that there is an immediate danger of frequency coalescence for values of $K_\theta > 1.0$. However, there is the recognized danger that a large value of K_θ would cause an early frequency coalescence of the modes comprising predominantly fundamental bending and predominantly second bending.

It is clear that the region of instability discussed in the preceding paragraph may be eliminated by proper choice of K_θ and ξ_G. On the other hand, the lower-frequency curve becomes zero at the same value of $\bar{T}_0$ for all values of K_θ and ξ_G. At this point an unstable mode exists in which bending of the beam occurs. Since no choice of K_θ and ξ_G exists which will eliminate this instability (except $K_\theta = 0$, which reduces the bending participation to zero in this mode), the value of the thrust at this crossing point is considered to be the critical thrust magnitude. This assumes that the higher-frequency curves not shown in Fig. 4 are well behaved up to this point.

Silverberg[5] showed that for a uniform free-free beam with an end thrust constrained to move parallel to a fixed line the magnitude of the thrust at which a buckled equilibrium shape becomes possible is determined from the equation $J_{2/3}(\frac{2}{3}\bar{T}_0^{1/2}) = 0$, where $J_{2/3}$ is the Bessel function of the first kind, of order $\frac{2}{3}$. From this equation it is found that

$$\bar{T}_{cr} = 25.67 \tag{23}$$

The general shape of the beam in this buckled position is sketched in Fig. 5. The inertia forces shown acting on the beam are in equilibrium with the thrust T_{cr}. Configuration A

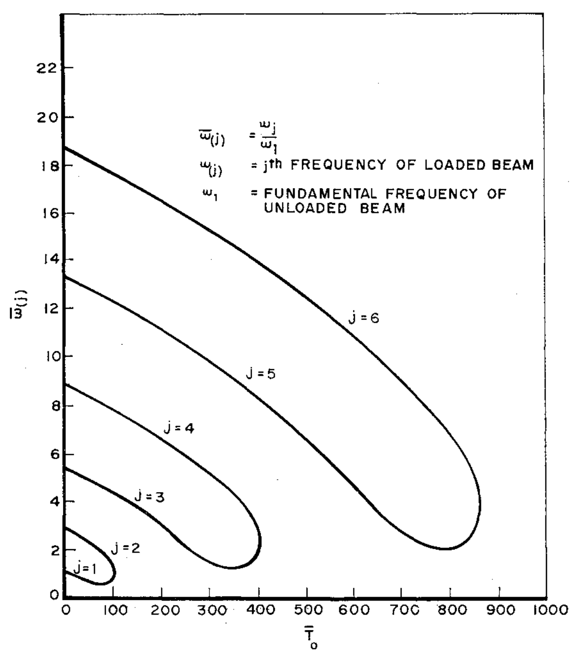

Fig. 3 Coalescence of higher mode frequencies ($N = 8$).

represents the system analyzed by Silverberg. Equilibrium is preserved when the system undergoes a simple rotation to produce the configuration represented by B. It is clear that the amount of rotation required to rotate from A to B may be adjusted to satisfy the relationship $\theta = K_\theta \psi_G$, in which case configuration B represents a possible zero-frequency configuration for the case of the beam with arbitrary feedback control.

We conclude, therefore, that a zero-frequency solution will exist for arbitrary values of K_θ (nonzero) and ξ_G at a thrust given by Eq. (23). The discrepancy between the critical thrust given by this equation and the point at which the frequency curves cross the $\bar{\omega} = 0$ axis in Fig. 4 is due to the fact that only two bending degrees of freedom were used in calculating the curves of the figure. Computations were also made (although the results are not shown in the figure) using three bending degrees of freedom. These analyses showed the crossing point to be at $\bar{T}_0 = 26.08$, which differs from the value given in Eq. (23) by approximately 1.5%.

Application of Results to Missile Stability

Virtually all current missiles have some form of directional control system. Thus, for missile application, the value $\bar{T}_0 = 25.67$ is more significant as a critical thrust than is the value at which the two lowest bending frequencies coalesce. Use of angular-rate feedback control would not affect this critical thrust, since the rotational velocity is zero in the displaced equilibrium position at the critical thrust magnitude.

We may attempt to compare the critical thrust for a uniform beam with the thrust on a missile in free flight. However, since most missiles have highly nonuniform characteristics, such a comparison is at best only an approximation. From the equation $\bar{T}_0 = mla$ (where a is the longitudinal acceleration) and from the expression for $\lambda_1{}^4$ in Eq. (9), we obtain

$$\bar{T}_0 = \lambda_1{}^4 a/\omega_1{}^2 l \tag{24}$$

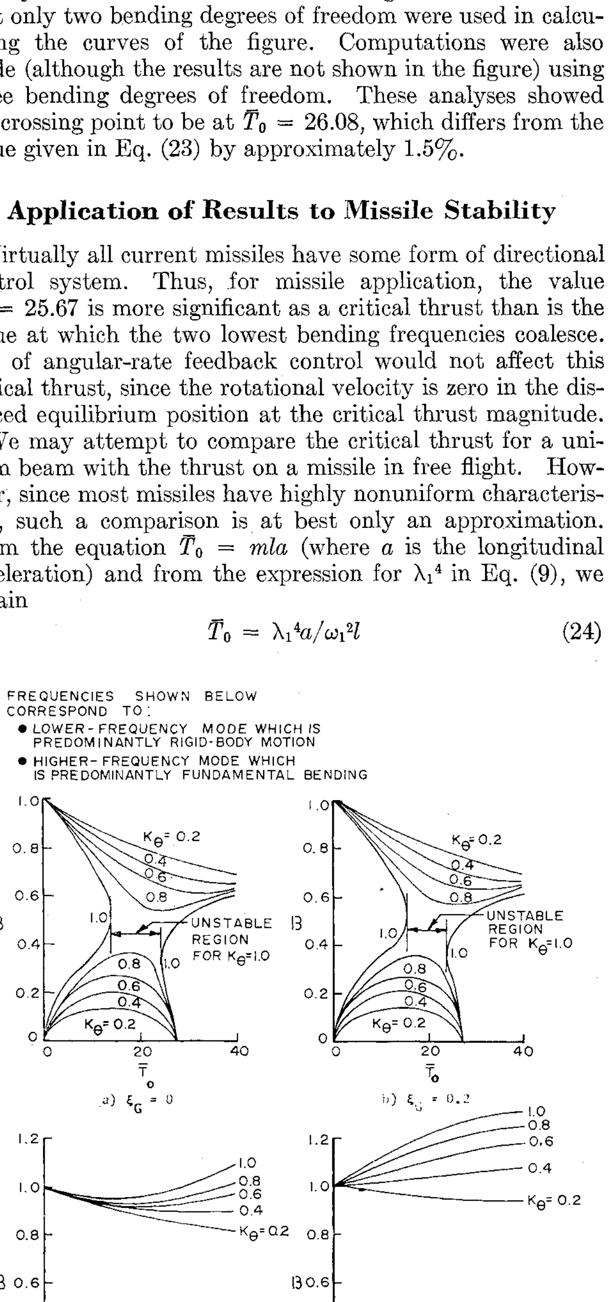

Fig. 4 Variation of frequency with thrust for a range of values of K_θ and ξ_G ($N = 2$).

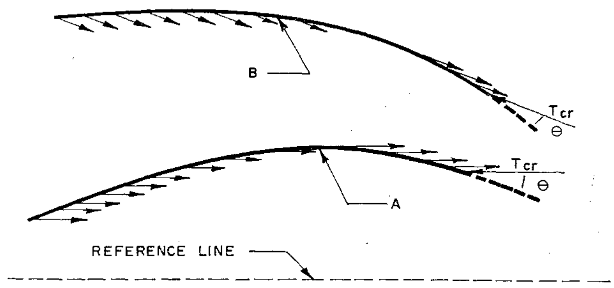

Fig. 5 Buckled equilibrium shape.

Then, assuming $\bar{T}_0 = 25.67$ and introducing g, the acceleration due to gravity at the earth's surface, as a nondimensionalizing factor, we obtain

$$\frac{a}{g} = \frac{25.67\omega_1{}^2 l}{500.6g} \tag{25}$$

If we assume that Eq. (25) may be applied in an approximate way to a nonuniform missile, where ω_1 is the fundamental frequency in bending (radians per second) and l is the length of the missile, we may obtain an estimate of the acceleration required to cause the missile to become unstable. Assuming $\omega_1 = 2$ rad/sec, and $l = 300$ ft (possible values for a large missile on an escape mission), and using $g = 32.2$ ft/sec^2, we obtain $a/g = 19.1$, which is an acceleration four to five times greater than that normally experienced by modern missiles. Thus, it is not anticipated that an instability of this type will be experienced by current missiles. However, it is conceivable that future space vehicles, perhaps assembled in space with extremely flexible structures, will be faced with such problems.

Part II: Stability under Pulsating Thrust

We now turn to the more general problem in which the periodically varying thrust component is retained. In this case we anticipate a parametric instability in which frequency ratios become important in addition to thrust amplitudes. Several methods have been developed for determining the stability of a system of equations of the type given in Eq. (19). For the purpose of this investigation, however, an extension of Hill's method[6] of infinite determinants is used. The derivation of the extended theory is quite lengthy and is being submitted for publication in a separate paper. Thus, in the present paper little space is devoted to the method of analysis; primarily, our attention will be directed toward a presentation of the results of the analysis.

For the sake of continuity a brief description of the method used to determine the stability of Eq. (19) is given. It is known[7] that solutions to equations of this type may be expressed in the form

$$[q_k] = e^{i\alpha\bar{\Omega}\tau}\,[\Psi_k(\tau)] \tag{26}$$

where α is called the characteristic constant and $[\Psi_k]$ is a column matrix in which each element has a periodic variation of period $2\pi/\bar{\Omega}$. This is the period of variation of the varying thrust component in terms of the nondimensional time variable τ.

Expanding $[\Psi_k]$ in a complex Fourier series

$$[\Psi_k] = \sum_{m=-\infty}^{\infty} [c_k]^{(m)} e^{im\bar{\Omega}\tau} \tag{27}$$

we may write (26) as follows:

$$[q_k] = \sum_{m=-\infty}^{\infty} [c_k]^{(m)} e^{i(\alpha+m)\bar{\Omega}\tau} \tag{28}$$

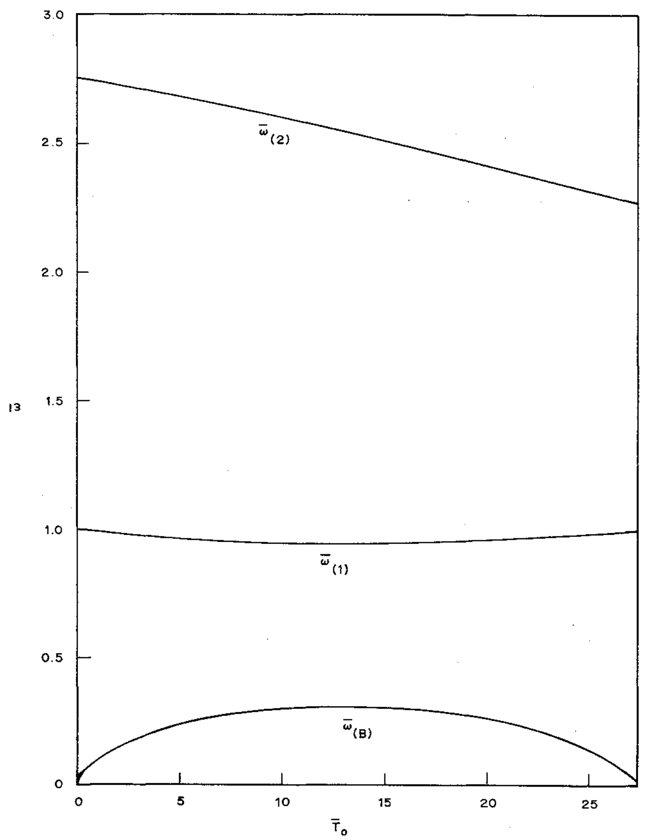

Fig. 6 Variation of frequencies with thrust ($N = 2$, $K_\theta = 1.0$, $\xi_G = 0.5$).

where $[c_k]^{(m)}$ is a column matrix of constants, the kth element in the mth matrix being denoted by $c_k^{(m)}$.

If we now substitute the series solution of $[q_k]$, as given in Eq. (28) into Eq. (19) and cancel the common factor $e^{i\alpha\bar{\Omega}\tau}$, we obtain

$$-\bar{\Omega}^2 \sum_{m=-\infty}^{\infty} [c_k]^{(m)}(\alpha + m)^2 e^{im\bar{\Omega}\tau} + [F_{jk}] \sum_{m=-\infty}^{\infty} [c_k]^{(m)} \times e^{im\bar{\Omega}\tau} + \frac{\gamma}{2}(e^{i\bar{\Omega}\tau} + e^{-i\bar{\Omega}\tau})[G_{jk}] \sum_{m=-\infty}^{\infty} [c_k]^{(m)} e^{im\bar{\Omega}\tau} = 0 \quad (29)$$

in which $\cos\bar{\Omega}\tau$ has been expressed in the form $\cos\bar{\Omega}\tau = \frac{1}{2}(e^{i\bar{\Omega}\tau} + e^{-i\bar{\Omega}\tau})$.

Note that if $(e^{i\bar{\Omega}\tau} + e^{-i\bar{\Omega}\tau})$ is combined with $e^{im\bar{\Omega}\tau}$, every term in Eq. (29) has a time-varying factor $e^{ik\bar{\Omega}\tau}$, k an integer. Equation (29) can be satisfied for all values of τ only if the collected coefficients of like exponentials are individually equal to zero. Thus, it is required that the following equations hold:

$$-\bar{\Omega}^2(\alpha + m)^2[c_k]^{(m)} + [F_{jk}][c_k]^{(m)} + \frac{\gamma}{2}[G_{jk}]\{[c_k]^{(m-1)} + [c_k]^{(m+1)}\} = 0$$

$$m = \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \quad (30)$$

Equation (30) represents a set of linear, homogeneous, algebraic equations in the unknowns $c_k^{(m)}$, $k = 1, 2, \ldots, N + 1$, $m = \ldots -3, -2, -1, 0, 1, 2, 3, \ldots$. In order that such a system of equations has nontrivial solutions, the determinant of coefficients must be equal to zero. Since the index m is unbounded, the order of the determinant is infinite; however, convergence is insured by division of Eq. (30) by the appropriate factors. The requirement that the determinant vanish provides a criterion for evaluating α. It may be shown that the characteristic equation resulting from this requirement is expressible in the form of a polynomial of order $N + 1$ in $\cos 2\pi\alpha$. The stability of the system is then determined by solution of the roots $Z = \cos 2\pi\alpha$ of this polynomial. From these roots the corresponding values of α are readily obtained from the equation[8]

$$\alpha = -(i/2\pi)\ln[Z \pm (Z^2 - 1)^{1/2}] \quad (31)$$

Designating

$$R_1 e^{i\beta_1} = Z + (Z^2 - 1)^{1/2} \quad (32)$$

$$R_2 e^{i\beta_2} = Z - (Z^2 - 1)^{1/2} \quad (33)$$

we see that, by adding and subtracting Eqs. (32) and (33) and then squaring the resulting equations, we arrive at the following equations:

$$R_1^2 e^{2i\beta_1} + 2R_1R_2 e^{i(\beta_1+\beta_2)} + R_2^2 e^{2i\beta_2} = 4Z^2 \quad (34)$$

$$R_1^2 e^{2i\beta_1} - 2R_1R_2 e^{i(\beta_1+\beta_2)} + R_2^2 e^{2i\beta_2} = 4(Z^2 - 1) \quad (35)$$

Subtracting Eq. (35) from (34), we obtain

$$R_1R_2 e^{i(\beta_1+\beta_2)} = 1 \quad (36)$$

from which it follows that

$$R_1R_2 = 1 \quad (37)$$

$$\beta_1 = -\beta_2 \quad (38)$$

Substituting Eq. (32) into Eq. (31), we find that

$$\alpha_1 = \frac{\beta_1}{2\pi} - i\frac{\ln R_1}{2\pi} \quad (39)$$

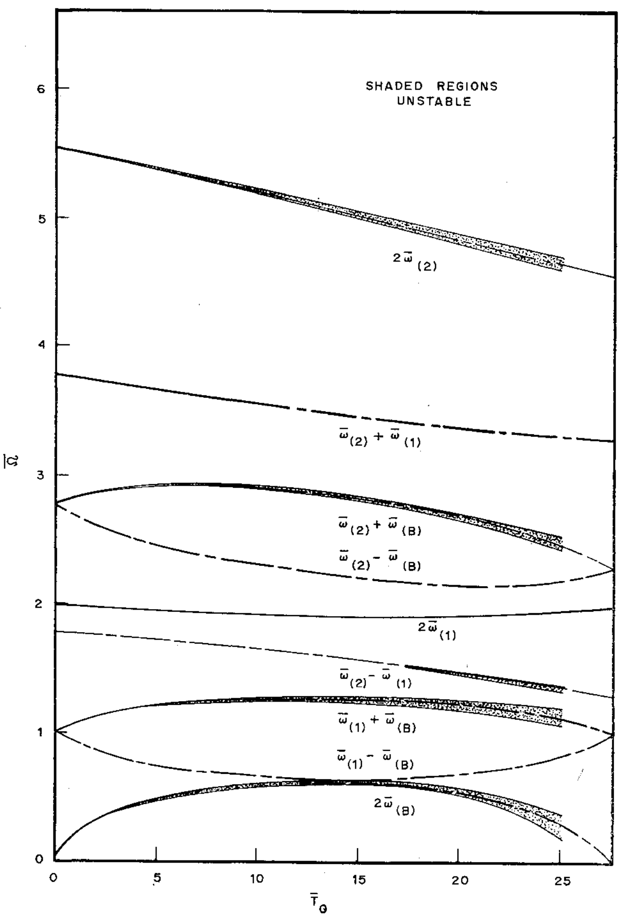

Fig. 7 Stability diagram ($\bar{\omega}_L = 100$, $N = 2$, $\gamma = 0.1$, $K_\theta = 1.0$, $\zeta_G = 0.5$).

$$e^{i\alpha_1\bar{\Omega}\tau} = \exp\left[\frac{\ln R_1}{2\pi}\bar{\Omega}\tau\right]\exp\left[i\frac{\beta_1}{2\pi}\bar{\Omega}\tau\right] \qquad (40)$$

The nonoscillatory factor $\exp[(\ln R_1/2\pi)\bar{\Omega}\tau]$ determines whether the solution (26) will converge or diverge. If $R_1 > 1$, the solution is divergent; if $R_1 \leq 1$, the solution is stable. Observe, however, from Eq. (37), that unless $R_1 = R_2 = 1$ at least one root must be divergent (it is required with no restriction that R_1 and R_2 be real and positive).

We conclude that, unless R_1 and R_2 are both equal to unity, an unstable solution exists. It follows that all α must be real in order that the system be stable. Therefore, for stability, it is necessary that all roots Z of the characteristic equation be real and limited to the range $-1 \leq Z \leq 1$.

Mettler has classified the areas of instability as type-1 and type-2 regions.[7] The boundaries of the type-1 instability regions are characterized by values of $Z = \pm 1$. On the boundaries of the type-2 regions, incipient complex solutions exist, characterized by double roots of the characteristic polynomial. Furthermore, Mettler shows that, as γ approaches zero, the type-1 instability regions impinge upon the loci $\bar{\Omega} = 2\bar{\omega}_{(i)}/m$ (m a positive integer), whereas the type-2 regions impinge upon the loci $\bar{\Omega} = |\bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}|/m$. The frequency parameters $\bar{\omega}_{(i)}$ and $\bar{\omega}_{(j)}$ are frequencies of the system nondimensionalized by division by the frequency ω_1. The most significant regions of instability correspond to values of $m = 1$, and the numerical investigations presented in this paper will be limited to such cases.

Presentation of Results

Values of $K_\theta = 1.0$ and $\xi_G = 0.5$ were arbitrarily chosen for a sample study. A parametric study showing the effect of values of K_θ and ξ_G other than those selected was not made, since such a study would have no significant qualitative value. The only anticipated variation would be a change in the location of the unstable regions due to a change in the vibration frequencies themselves. A value of $\gamma = 0.1$ was assumed for all cases presented here. A number of cases were run for a value of $\gamma = 0.5$ also; the results of these investigations indicated that the widths of the unstable regions were approximately linear with γ (at least in the range below $\gamma = 0.1$). However, the results of these studies are not shown here.

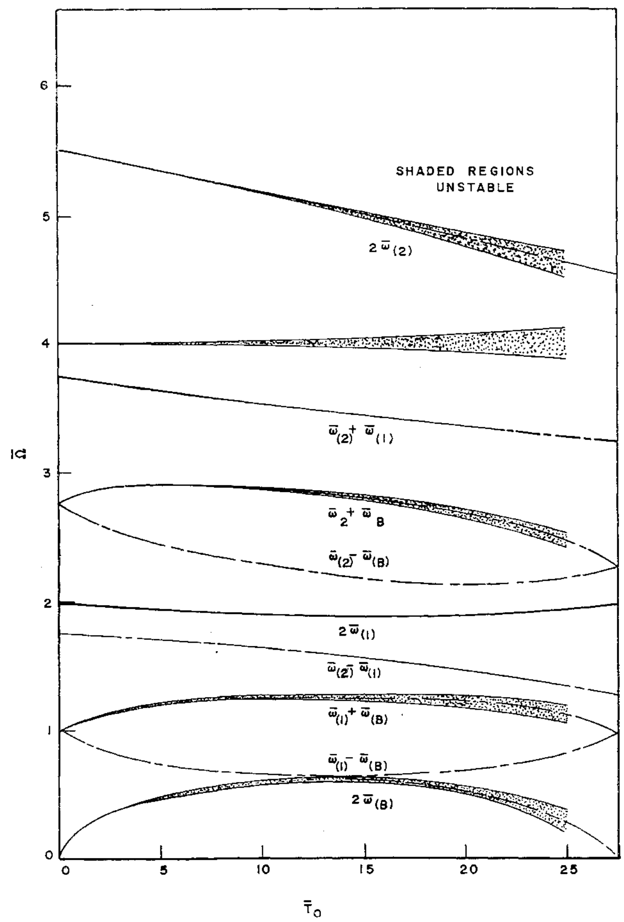

Fig. 8 **Stability diagram ($\bar{\omega}_L = 4.0$, $N = 2$, $\gamma = 0.1$, $K_\theta = 1.0$, $\zeta_G = 0.5$).**

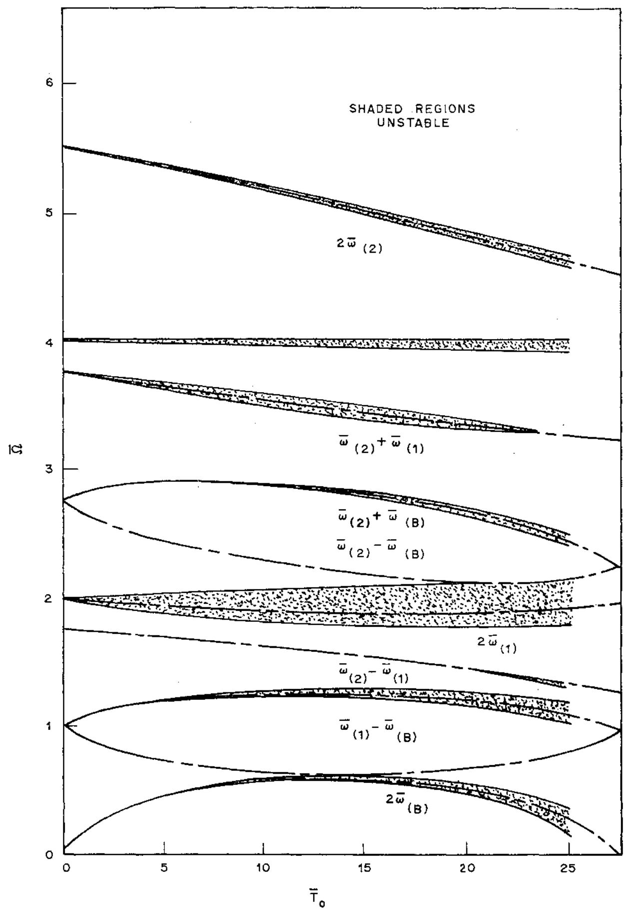

Fig. 9 Stability diagram ($\bar{\omega}_L = 2.0$, $N = 2$, $\gamma = 0.1$, $K_\theta = 1.0$, $\zeta_G = 0.5$).

Two bending degrees of freedom were assumed in addition to the rigid-body degrees of freedom. After eliminating the rigid-body translation coordinate q_A, three coordinates, q_B, q_1, and q_2, remain. Thus the matrix $[q_k]$ of Eq. (19) is

$$[q_k] = \begin{pmatrix} q_1 \\ q_2 \\ q_B \end{pmatrix} \qquad (41)$$

considering that a larger number of bending degrees of freedom would improve the accuracy but increase the difficulty of the computations. Qualitatively, it is not expected that the results would be greatly changed.

The curves of Fig. 6 show the variations of the frequencies as the thrust T_0 is increased. The frequencies are designated $\bar{\omega}_{(B)}$, $\bar{\omega}_{(1)}$, and $\bar{\omega}_{(2)}$, so named because of the predominance of motion in the coordinates q_B, q_1, and q_2, respectively. It is noted that the frequency $\bar{\omega}_{(B)}$ is reduced to zero at a value of $\bar{T}_0 = 27.5$. The thrust parameter $\bar{T}_0$ will be limited to values less than this critical value.

Regions of instability are expected when Ω is in the vicinity of either twice the frequencies, or the sum or difference of any two of the frequencies shown in Fig. 6. The unstable regions actually computed for these vicinities are shown in Fig. 7 for a value of $\bar{\omega}_L = 100.0$, a value which, for the purposes of this investigation, is essentially infinite (i.e., the beam is longitudinally very stiff). In some cases no instabilities were discovered in the vicinities of expected unstable regions. Such cases are indicated by a dashed line. In other cases instabilities were found, but the regions were so narrow that they only appear as solid curves in the figure (note, for example, $\bar{\Omega} = 2\bar{\omega}_{(1)}$). In one case ($\bar{\Omega} = \bar{\omega}_{(2)} + \bar{\omega}_{(1)}$), a narrow unstable region disappears completely beyond an intermediate value of $\bar{T}_0$. In another case ($\bar{\Omega} = \bar{\omega}_{(2)} - \bar{\omega}_{(1)}$), although no instabilities were found for the smaller values of $\bar{T}_0$, a narrow region developed beyond an intermediate value of $\bar{T}_0$.

The unstable regions for a longitudinal frequency $\bar{\omega}_L = 4.0$ are shown in Fig. 8. The most significant difference between this case and the previous case is the presence of a new region of instability in the vicinity of $\bar{\Omega} = 4.0$.

The unstable regions for $\bar{\omega}_L = 2.0$ are shown in Fig. 9. In this case, an unstable region in the vicinity of $\bar{\Omega} = 2.0$ appears to merge with the region expected in the vicinity of $\bar{\Omega} = 2\bar{\omega}_{(1)}$. A relatively broad region of instability results. Additionally, an unstable region appears in the vicinity of $\bar{\Omega} = 4.0$, which is twice $\bar{\omega}_L$. Although not shown here, unstable regions also appear to exist in the vicinity of $\bar{\Omega} = 6.0, 8.0$, etc.

Conclusions

It has been shown that a rocket vehicle with an end thrust of magnitude $T_0 + T_1 \cos\Omega t$ may exhibit instabilities in its bending modes for any range of values of the thrust T_0. When the vehicle is assumed to be very stiff longitudinally, unstable solutions were found to exist for frequencies of thrust variation in the vicinity of 1) twice any of the bending frequencies, 2) the sum of any two of the bending frequencies, or 3) the difference of any two of the bending frequencies. When the longitudinal compliance of the vehicle is taken into consideration, it was found that instabilities also occur for frequencies of the thrust variation in the vicinity of the longitudinal natural frequencies. These instabilities are expected to be most severe when the fundamental longitudinal frequency is itself in the vicinity of one of the already critical regions.

It is concluded that the existence of parametric instabilities due to periodic variations of the thrust magnitude is a definite possibility in modern missiles. Because of the proximity of the fundamental longitudinal and fundamental bending frequencies (at least in certain missiles), it is apparent that the longitudinal compliance of the missile may play a significant role in these instabilities. Although the magnitude of the thrust ratio γ will probably not be as large for an actual missile as the values considered here ($\gamma = 0.1$ was assumed for the analyses conducted herein), it is possible that such values will be sufficiently large (of order of magnitude 1 or 2%) that instabilities may develop which would not be overcome by structural and control system damping.

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